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Modeling a Professional Driver in Ultra-High Performance Maneuvers with a Hybrid Cost MPC

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Abstract— **During racing or vehicle performance evaluation, a professional driver looks ahead and chooses the vehicle controls to best maneuver the circuit or track in minimum time. This process is carried out in a receding horizon fashion around the circuit or track in a process that closely resembles Model Predictive Control (MPC). Professional drivers quickly build mental models of the circuit or track of how best to exploit its features in order to negotiate the full maneuver in minimum time. In this article, we implement a method for exploiting future preview information into a local solution of a MPC which indirectly models a driver learning a particular circuit or track. A hybrid cost MPC structure that is capable of switching between two different driving styles is used to achieve the incorporation of this future preview information. It will be shown that this future preview information (beyond the local MPC horizon) can help to alleviate some of the sub-optimal behavior inherent in MPC. The concept is then demonstrated on a chicane maneuver. The proposed scheme is compared against the exact time-optimal control solution for this maneuver.**

I. INTRODUCTION

Modeling lap time performance has been a rich topic of research dating back as early as the 1930's where [1] credits Mercedes Benz with the first known application of engineering tools used to model a race car negotiating a track for minimum time. In the beginning, the driver's influence was neglected all together and the steady state performance of the vehicle was all that was used to calculate lap time. As research progressed, it was clear that the driver and transient vehicle performance have a large influence on the problem of minimum time maneuvering.

This led to the two stage optimization where a racing line is first computed based on geometric optimization (maximizing radius of curvature or minimizing distance travelled) [2], [3], evaluation of a vehicle model and optimizing a parameterized curve for the racing line [4], or even driving telemetry. Once the path has been determined, a driver will attempt to track the desired trajectory in minimum time. Techniques for this later stage are widely varying; however, optimal path preview techniques rooted in the work [5] and later refined in [6] are commonplace. Following a predefined path in minimum time is a typical problem found in robotics and [7] shows how the problem can be converted into a convex optimization. This technique was expanded in [8] for for simulations of vehicles. This approach is still widely used for application where realtime control (autonomous driving) is paramount [9] and consists of an offline path planning and an online path following stage.

An alternative to the two stage approach is combining path planning and path following. This strategy assumes the driver is a pure time-optimal controller and seeks to find the optimum set of controls to minimize the objective (lap time) subject to vehicle dynamic constraints and path constraints. This approach dates back to the work in [10] which can be considered the first significant formulation. Once posed as a time-optimal control, two key solution techniques are typically used. Indirect methods involve a solution to first order necessary conditions via application of Pontryagin's Minimum Principle and deriving the adjoint system equations. The indirect methods lead to two-point boundary value problems [11] which can be quite arduous to solve for complicated dynamics and constraints. Alternate solution techniques are direct methods which aim to solve the optimal control problem itself by transcribing the continuous optimal control problem into a nonlinear programming problem (NLP). Direct methods are typically used when solving minimum time vehicle maneuvering problems and these numerical techniques are reviewed in [12], [13].

To facilitate computations, an arbitrarily long maneuver such as a racing circuit is typically broken down and solved recursively over short segments either by multiple shooting techniques [14] or Model Predictive Control (MPC) [15], [16]. Orthogonal collocation techniques have furthered the state of the art and a full racing circuit has been solved via these techniques as shown in [17]. This topic has been extended to include some key high fidelity effects such as modeling additional vehicle control systems such as energy recovery systems [18], a thermally dependent tire model [19], and three dimensional road effects [20].

Most of the reviewed literature is geared towards time-optimal driving (i.e., finding the absolute minimum maneuvering time) without considering how a real human driver compares to an optimal controller. The analogy of assuming that a driver is a true optimal controller with full knowledge of all future path preview information is that a driver chooses his/her control action on the start line based on how he/she will cross the finish line. While a professional driver can indeed learn a particular race circuit, it is more likely that a driver will look ahead a particular preview distance and choose his or her actions in order to best exploiting the learned track features (learned through exploratory and warmup laps) in a similar process to MPC. Modeling such nuances of human drivers is not a new concept. In fact,

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[21] shows that realistic human control bandwidth limits achievable maneuvering time.

The work in [22] attempts to address this problem with robust optimal control. The work in [23], [24] shows how boundary conditions and cost functions can be altered to reproduce advance driving techniques. Although not strictly motivated by minimum time maneuvering, [25] shows how varying cost functions can represent different driving styles. These literatures suggest that a true human driver model is not merely a deterministic optimal controller with a fixed cost function. Experience in testing and racing has shown two different driving styles can lead to nearly identical lap times.

In this paper, we propose to model a professional driver with a hybrid cost MPC structure where the local/preview horizon cost function is switched to approach minimal time maneuvering for the track or circuit. We assume the driver has two local modes, one that approximates minimum-time driving and another that approximates maximum-velocity driving. The driving style of maximizing-velocity is motivated by the fact that MPC-based modeling does not consider the entire circuit's features and in some cases the MPC is unaware that it is more appropriate to sacrifice time in a particular short segment and maximize velocity in order to minimize the complete maneuvering time. The hybrid cost switching allows future path preview information to be incorporated into the local MPC solution and some of the sub-optimal properties of a fixed-cost MPC are alleviated. We conjecture that a driver is able to change his/her driving style around a particular circuit to best exploit its features and minimize total maneuvering time.

This paper is organized as follows. In section II, the vehicle model utilized in this research is described. Section III includes details on the optimal control strategies that were adopted to model the human driver and the reference time-optimal solution. Section IV presents results obtained via the different control strategies discussed in Section III. Finally, conclusions are offered in Section V.

II. VEHICLE MODEL

For the purposes of this paper, a simple particle motion vehicle model that is widely used in the area of autonomous vehicles is employed [26]. This simple vehicle description is capable of capturing first order vehicle dynamics in a computationally efficient formulation. The vehicle consists of a non-holonomic description of the center of gravity (CG) motion constrained by a friction ellipse. The inputs u_1, u_2 represent the driver's longitudinal and lateral control. These inputs are then passed through first order lags as seen in (1d,1e) and these equations are used to define the vehicle's longitudinal and lateral accelerations (a_t, a_n respectively). The constants τ_{a_n}, τ_{a_t} are used to approximate the vehicle dynamics.

The vehicle's coordinate system is written in path intrinsic coordinates, i.e., the planar motion is described as a lateral deviation from the path centerline (e_y) and heading angle deviation (e_ψ) which is defined as the difference between the path heading angle (ψ_s) and the vehicle heading angle (ψ_v). This coordinate system allows for a convenient description of the road width constraints ($\underline{e_y} \leq e_y \leq \overline{e_y}$). The path itself is defined from the path curvature ($\kappa(s)$) defined along the path's curve length often called station s.

The quantity v_t represents the vehicle's velocity at its CG while $\dot{s}$ represents the vehicle's velocity along the path and will be used later to transform the independent variable from time to distance travelled along the path. The vehicle's motion can be seen in Fig. 1 and is defined as follows:

$$\dot{v_t} = a_t \tag{1a}$$

$$\dot{e}_\psi = \frac{a_n}{v_t} - v_t cos(e_\psi)\frac{\kappa(s)}{1 - e_y\kappa(s)} \tag{1b}$$

$$\dot{e}_y = v_t sin(e_\psi) \tag{1c}$$

$$\dot{a_t} = 1/\tau_{a_t}\,(u_1 - a_t) \tag{1d}$$

$$\dot{a_n} = 1/\tau_{a_n}\,(u_2 - a_n) \tag{1e}$$

$$\dot{t} = 1 \tag{1f}$$

$$\dot{s} = \frac{v_t cos(e_\psi)}{1 - \kappa(s)e_y} \tag{1g}$$

For compactness, we define $\mathbf{x} = \begin{bmatrix} v_t & e_\psi & e_y & a_t & a_n & t \end{bmatrix}^T$ as the state vector and $u \in \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ as the control vector. As is commonly done in the minimum maneuvering time literature, the system is transformed from the independent variable time to path distance traveled via the transformation shown in (2). This way, the final value of the independent variable (in our transformed system, path distance travelled) is a fixed-value versus a parameter that must be optimized (as is the case in the free final time problem).

$$\frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \frac{\dot{\mathbf{x}}}{\dot{s}} = f(x(s), u(s), s) \tag{2}$$

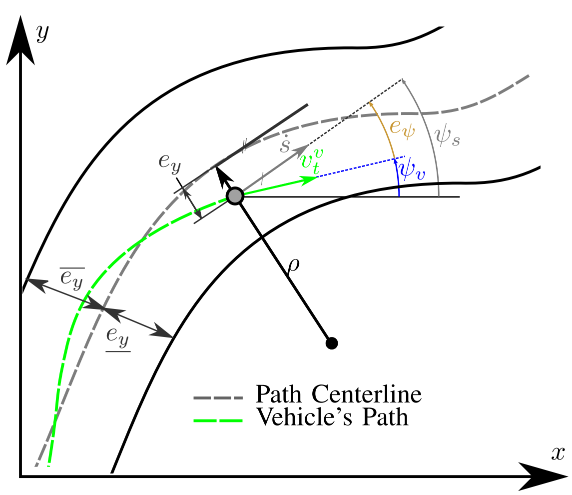

Fig. 1. Particle motion model path intrinsic coordinates. Note that the instantaneous curvature $\kappa = 1/\rho$ is used in the formulation to alleviate singularities found on a straight road where the radius of curvature (ρ) tends toward infinity.

III. OPTIMAL CONTROL MODELS

In general, an optimal control problem seeks to find the state $x \in \mathbb{R}^n$ and control input $u \in \mathbb{R}^m$ that minimize a cost functional. A general form of this cost, commonly referred to as the problem of Bolza, can be written as:

$$J = \Phi(s_0, x(s_0), s_f, x(s_f)) + \int_{s_0}^{s_f} \mathcal{L}(s, x(s), u(s)) ds \quad (3)$$

For the purposes of this work, a sufficiently general optimal control problem may be posed as:

$$\begin{array}{ll} \min\limits_{u} & J \\ s.t. & \frac{d\mathbf{x}}{ds} - f(s, x(s), u(s)) = 0 \\ & g(s, x(s), u(s)) = 0 \\ & h(s, x(s), u(t)) \leq 0 \\ & g_b(x(s_0), x(s_f), u(s_o), u(s_f)) = 0 \end{array} \quad (4)$$

Where $f(\cdot) \in \mathbb{R}^n$ represents the system dynamics described by (2). The function $g(\cdot) \in \mathbb{R}^{n_g}$ and $h(\cdot) \in \mathbb{R}^{n_h}$ represent the equality and inequality constraints, respectively. The function $h(\cdot)$ will be used to place constraints on the friction ellipse ($a_n^2 + a_t^2 - a_{max}^2 \leq 0$) and the lateral deviation of the vehicle is bounded ($\underline{e_y} \leq e_y \leq \overline{e_y}$) to constrain the vehicle to operate within the predefined road width. The function $g_b(\cdot) \in \mathbb{R}^{n_{g_b}}$ refers to boundary constraints at the start (s_0) and finish (s_f).

A. Time-Optimal Solution - Reference Solution

The classic approach to the problem is to find the full optimal control solution over the complete maneuver that minimizes the cost function:

$$J_t = \int_{s_0}^{s_f} \frac{1}{\dot{s}} ds = t_f \quad (5)$$

In general, this problem is quite difficult to solve for arbitrary tracks; however, its solution offers a reference solution to the hybrid cost MPC which will be described later. For our purposes, the optimal control problem in (6) is solved via orthogonal collocation methods (implemented in the software package GPOPS-II [27]).

$$\begin{array}{ll} \min\limits_{u} & J_t \\ s.t. & \frac{d\mathbf{x}}{ds} - f(s, x(s), u(s)) = 0 \\ & a_n^2 + a_t^2 \leq a_{max}^2 \\ & \underline{e_y} \leq e_y \leq \overline{e_y} \end{array} \quad (6)$$

B. MPC Approach

Model Predictive Control (MPC) is a receding horizon approach that solves the full optimal control problem by recursively solving short preview segments and advancing forward (by a distance Δs) until the full solution is obtained (as seen in Fig. 2). Each segment is denoted with superscript i and recursively solved by N segments until the complete interval is solved. Because the MPC approach neglects information outside of the local short horizon (future preview information), the approach is inherently sub-optimal. The benefits to MPC, however, are numerical stability since the solution over a short segment is simpler than the full problem. Because of this simplicity, the problem typically results in significantly decreased computational cost. Also, if the cost function can be expressed as a problem of least squares (7) (a subset of the problem of Bolza), efficient algorithms exist for solving this problem [28]. Therein, a multiple shooting technique and Sequential Quadratic Programming (SQP) is used to solve the underlying NLP. Realtime nonlinear MPC (NPMC) that uses this algorithm has been realized in [29] for small remote controlled vehicles on a scale. In order to achieve a working NMPC controller, weights (P, Q, R) and reference trajectory (x_{ref}) for a least squares cost (J_{LSQ}) were tuned to approximate the true time-optimal problem posed in (5). In other words, J_{LSQ} was tuned such that $J_{LSQ} \approx J_t$. The least squares cost is defined as:

$$J_{LSQ} = \underbrace{\|x(s_h) - x_{ref}(s_h)\|_P^2}_{\text{Terminal Cost}} + \int_{s_0}^{s_h} \Big\{ \underbrace{\|x(s) - x_{ref}(s)\|_Q^2}_{\text{State Cost}} + \underbrace{\|u(s)\|_R^2}_{\text{Control Cost}} \Big\} ds \quad (7)$$

where s_h represents the distance of the MPC horizon. The full MPC problem can be posed as follows:

$$\begin{array}{ll} \min\limits_{u^i} & J_{LSQ}^i \\ s.t. & \frac{d\mathbf{x}}{ds} - f(s^i, x(s^i), u(s^i)) = 0 \\ & a_n^2 + a_t^2 \leq a_{max}^2 \\ & \underline{e_y} \leq e_y \leq \overline{e_y} \\ & x_0^i = x(s_o^i) \\ & s^i \in [s_0^i, s_0^i + s_h],\ i = 1, 2, ..., N \end{array} \quad (8)$$

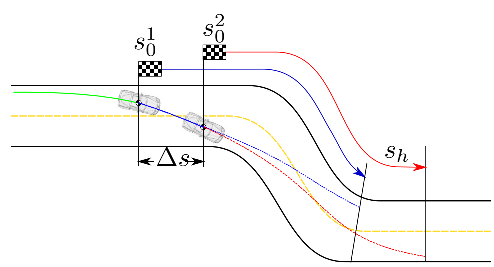

Fig. 2. MPC solution approach.

In the proceeding subsections (III-B.1 and III-B.2), the computationally efficient MPC framework will first be used to achieve an approximation of the driving styles representing both a time and velocity-optimal driver. Then, (in section III-B.3) a hybrid cost structure will be used to optimally switch between the time-optimal approximation and the velocity-optimal approximation.

1) Approximate Time-Optimal MPC: Due to the complex nature of the dynamics of the system, choices of MPC weights are critical. As previously mentioned, in order to utilize the efficient framework for solving the NMPC problem, the cost must be formulated in a least squares cost (7). Therefore the choice of weights $w = [P, Q, R]$ must be chosen such that $J_{LSQ} \approx J_t$. In [29], the weighs were tuned heuristically such that the weights on the terms involving the state time were much larger than all other weights; only small

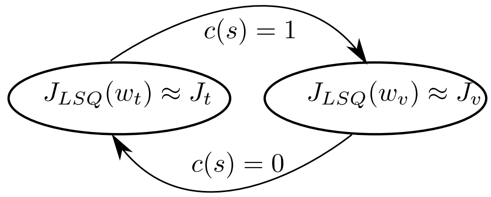

Fig. 3. Hybrid Cost MPC Structure.

weighting values were used as tracking penalties on the other states. This was used to improve numerical reliability. This led to a good approximation of time-optimal behavior. Rather than relying on heuristic tuning to find appropriate weights, the optimization problem posed in (9) was solved via a genetic algorithm to find the optimal set of weights. This solver, albeit computationally expensive, was a good choice of optimizer due to its tolerance of the highly sensitive nature of tuning these weights (w_t) by solving the optimization problem defined in (9). The weights w_t are bounded by lower and upper constraints, lb, ub respectively so that a reasonable result is achieved.

$$\begin{aligned} \min_{w_t=[P,Q,R]} \quad & J_t \\ s.t. \quad & \text{Sub MPC problem (8)} \\ & lb \leq w_t \leq ub \end{aligned} \tag{9}$$

2) Approximate Velocity-Optimal MPC: Again, optimization will be used to find the most appropriate weights for our MPC problem; however, now, the goal is to represent a driver maximizing velocity everywhere on the track. This is similar to the problem posed [23] where the objective was to maximize the corner exit velocity $v_t(s_f)$ to reproduce a particular driving technique prevalent in rally racing. For our velocity-optimal driver, the MPC weights (w_v) were tuned by solving (10) such that $J_{LSQ} \approx J_v = -||v_t||^2$. As with the time-optimal approximation, the weights (w_v) are bounded by lb, ub.

$$\begin{aligned} \min_{w_v=[P,Q,R]} \quad & J_v = -||v_t||^2 \\ s.t. \quad & \text{Sub MPC problem (8)} \\ & lb \leq w_v \leq ub \end{aligned} \tag{10}$$

3) Hybrid Cost MPC Driver: With good choices of weights approximating a time-optimal (w_t) and velocity-optimal (w_v) driver in this framework, it is desired to find the optimal switching between operating modes (as shown in Fig. 3 to minimize the total maneuvering time. For this, a final optimization problem can be posed to find the optimal switching (c) between driving styles. The decision variable, c, is a binary vector such that $c \in \mathbb{Z}_2^N,\ c^i \in \{0,1\},\ i = 1,2...,N$ where each MPC horizon's (denoted as horizon i) cost is defined as: $J_{LSQ}^i = \left\{(1-c^i)\cdot J_{LSQ}(w_t)\right\} - c^i \cdot J_{LSQ}(w_v)$ to facilitate the switching between driving styles over each horizon. In this optimization, the cost again is the true time-optimal cost (5) which was the same as the problem of finding weights for the time-optimal MPC approximation; however, it will be shown in the results, that this hybrid structure allows for a better maneuvering time than by adjusting weights alone. The problem posed in (11) is a mixed-integer programming problem, and a genetic algorithm will be used to find the appropriate sequence of driving styles c at each location of the track. When the optimal switching rule $c^{i^*} = 0$, the time-optimal approximation weights (w_t) are used in that local MPC solution and when $c^{i^*} = 1$, the velocity-optimal weights (w_v) are used. This switching is also depicted in Fig. 3.

$$\begin{aligned} \min_{c} \quad & J_t \\ s.t. \quad & \text{Sub MPC problem (8)} \\ & c \in \mathbb{Z}_2^N,\ c^i \in \{0,1\},\ i = 1,2...,N \end{aligned} \tag{11}$$

This cascaded optimization results in an optimal switching rule ($c^*(s)$) which allows for the path information of the full maneuver to be incorporated into the local MPC segments and aims to represent a driver learning a particular track/circuit to best exploit its features to minimize overall maneuvering time. To summarize, the hybrid cost MPC algorithm first finds the optimum MPC weights w_t, w_v for each driving style and then performs a search over the whole maneuver to find the optimum switching rule c^* to minimize the total maneuvering time.

IV. Results

The preceding optimal control analysis was carried out for a chicane type maneuver and nominal vehicle with parameters described in Table II and the resulting trajectories can be seen in Fig. 4. A comparison of the velocity traces of the three different strategies (reference optimal control solution (6), MPC time-optimal approximation (9), and the hybrid cost MPC (11)) can be seen in Fig. 5. Note that the maneuver takes place on the interval $s \in [-50m, 750m]$, where as the "lap time" was calculated on the interval $s_{timed} \in [0m, 650m]$. This simulates a racing environment where vehicle getting up to speed before timing starts and allows the timing to stop before the maneuver does.

Fig. 5 shows the optimal switching rule between time and velocity-optimal driving styles which is the direct output of problem described in (11). The velocity trace and control histories are also shown in Fig. 5. It is clear that the control histories have regions where the MPC solution chatters (between ①,② and ③,④). This phenomena occurs in the regions of the road where the curvature is changing rapidly causing numerical issues with the MPC algorithm. In an effort to retain computationally expediency, the MPC algorithm only performs one quadratic approximation per MPC horizon. The work in [30] provides justification of why this is sufficient for most realtime NMPC applications; however, because the full underlying NLP is not completely solved, the first order necessary conditions may or may not be fully satisfied at each time step. Thus, infeasible solutions exist in these MPC results as seen in the friction circle utilization plot of Fig. 6a. The filtered content of the controller (shown as a solid line), however, correlates well with the optimal reference solution. Moreover, it does indeed show improvement on the traditional time-optimal MPC.

In order to further substantiate our claims, an alternate MPC solver, (GPOPS-II [27]) was used to solve both the true time and velocity MPC problems (versus the LSQ cost used in the previous solution). Because these results are orders of magnitude higher in computational cost, using the outer loop genetic algorithm to find the optimum switching rule was not in the scope of this work. In order to show the hybrid MPC cost can have improvements over the traditional time optimal MPC, an alternate heuristically derived switching rule can be used which will be referred to hereafter as the split time switching rule. For this switching rule, the split time of the time optimal and velocity optimal MPC solutions were compared and when the velocity optimal solution was gaining time, the switching was changed to velocity optimal, otherwise it remained time optimal. More formally posed, the split time rule can be written as:

$$\Delta t = t^*_{v_t}(s) - t^*_t(s), \quad c = \begin{cases} 1, & \text{if } \frac{\Delta t}{ds} \leq 0 \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

Clearly, there is no guarantee of optimality with this switching decision; however, it does demonstrate improvement over the time optimal MPC. Fig. 7 shows the velocity trace as well as the control inputs to the system as well as the split time switching rule. Fig. 6b shows the friction envelope constraints and that no violations exist since the full underlying NLP is completely solved in this approach.

The results from all of the aforementioned simulations can be found in Table I and demonstrate how a hybrid cost MPC strategy helps circumvent some of the sub-optimality inherent in an MPC solution. An improvement in maneuvering time of just under 0.5s was found using the optimally switched hybrid cost versus the fixed approximate time optimal MPC. In terms of minimum-time maneuvering problems, this is a substantial decrease in maneuvering time. Much like a professional driver that can quickly learn a circuit or a track with a few exploratory laps, the cascaded optimization structure learns where it is best to switch between driving styles c.

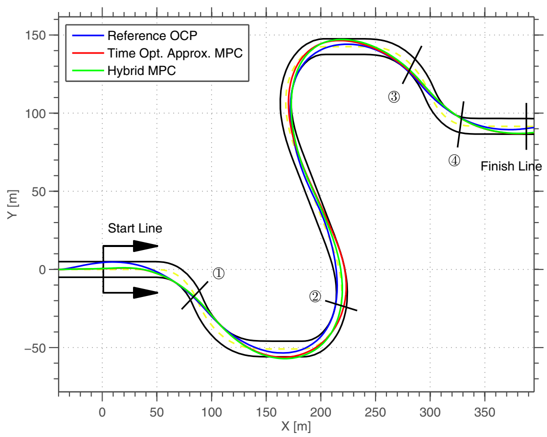

Fig. 4. Trajectories of the different control strategies.

V. CONCLUSIONS

In this paper, a hybrid cost MPC has been derived to alleviate some of the sub-optimality inherent in MPC for

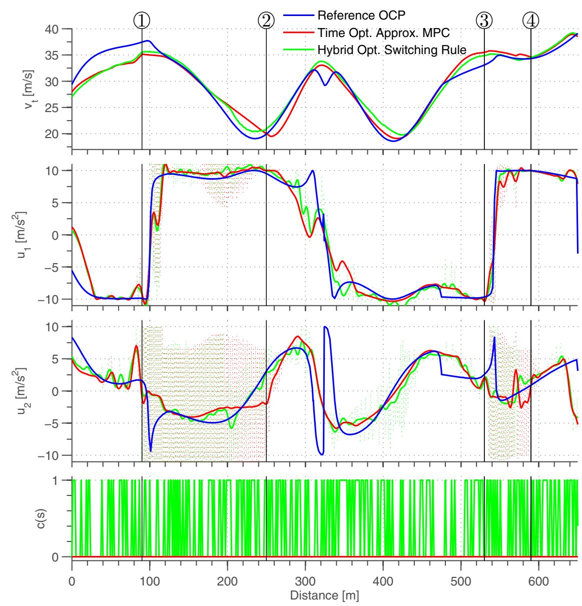

Fig. 5. Distance based histories of velocity, control inputs, and hybrid operating mode.

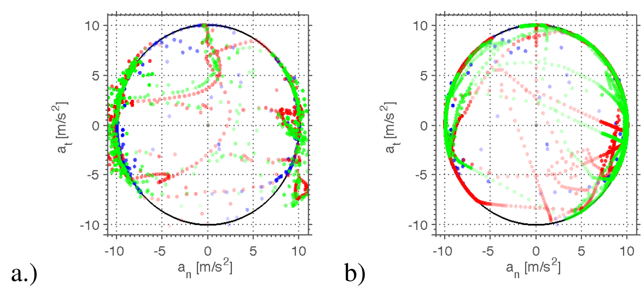

Fig. 6. Friction circle utilization plot. Blue - reference time optimal problem, Red - time optimal MPC, Green - hybrid MPC. a.) LSQ cost fast MPC solver, b.) True cost GPOPS-II MPC Solver.

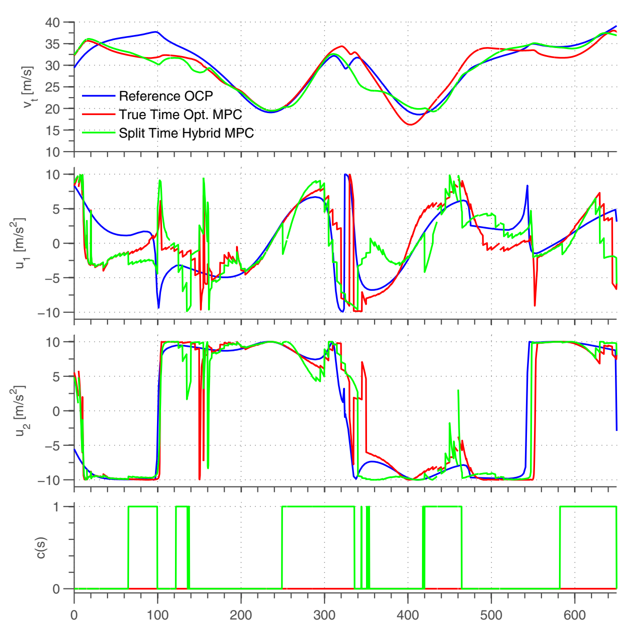

Fig. 7. Data traces from GPOPS-II MPC.

TABLE I

RESULTS FROM SIMULATION APPROACHES.

Simulation	Eqn.	Time [s]	Sub-optimality
Reference solution	(6)	21.977	0.00%
MPC time opt. approx.	(9)	22.787	3.69%
MPC velocity opt. approx.	(10)	22.887	4.14%
MPC hybrid c^*	(11)	22.349	1.69%
MPC True time opt.		22.698	3.28%
MPC hybrid split time c	(12)	22.591	2.79%

modeling drivers in ultra-high performance maneuvers. This was accomplished by a concatenation of MPC controllers representing driving styles of minimizing time and maximizing velocity over local segments. These controllers were optimally switched along the maneuver length to represent how a driver learning a particular circuit can exploit its features as best as possible. Simulations of the strategy applied to the chicane maneuver demonstrated that the hybrid cost MPC was advantageous to the fixed cost MPC in minimizing total time. This has broader implications in that if future information (beyond the scope of the local MPC segment) can be incorporated in the configuration of the local MPC controller, a better overall solution can be found through the full maneuver. The hybrid cost is only one vessel to accomplish this goal and future research will look into other possibilities for incorporating this information in the local MPC solutions.

APPENDIX

TABLE II

PARAMETERS USED IN SIMULATIONS.

Parameter	Description	Units	Value
τ_{an}	First order lag of the vehicle normal acceleration	s	0.075
τ_{at}	First order lag of the vehicle tangential acceleration	s	0.075
a_{max}	Bounding friction circle radius	m/s^2	10
N_u	Number of control actions over MPC horizon		75
s_h	MPC preview horizon	m	150

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